Affine Group Scheme Summary

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A List of Properties

- Linear
- Finite type
- Algebraic
- Connected
- Reduced
- Irreducible
- Integral
- Smooth
- Unipotent
- Solvable
- Reductive
- Semi-simple
- Algebraic Torus
- Split Algebraic torus
- Split
- Quasi-split
- Isotropic

A List of Distinguished Subgroups

- Derived Subgroup – (solvable)
- Unipotent Radical – (reductive)
- Solvable Radical – (semisimple)
- Maximal Torus – (split)
- Borel – (quasi-split)
- Neutral component
- Levi Subgroup
- Normalizer of torus
- Centralizer of torus
- Weyl group
- Parabolic
- Standard parabolic

We take the functor of points approach given in [Int]; it would proabably be a good idea to also go through [Mil]. We treat only of groups schemes over a field k. We denote the algebraic closre of k by $\bar{k}$ and the seperable closure k^{sep}.

An ***affine group scheme*** over k is a representable functor

$$G : \text{Alg}_k \to \text{Groups}$$

$$A \mapsto Hom_{Alg_k}(R, A)$$

The algebra that represents G is denoted $\mathcal{O}(G)$. Notice that there is an implicit condition that $Hom_{Alg_k}(R, A)$ actually forms a group for every A, which is not always the case for an arbitrary R, for instance the trivial k-algebra will have empty hom sets except for between itself, and the empty set is not a group.

am I right about this

The first example is the group scheme that sends an algebra to its multiplicative units. We denote this $\mathbb{G}_m$ this is represented by the k algebra

$$k[x,y]/(xy-1)$$

Another key example is GL_V sending an algebra A to the A-module automorphisms $V \otimes_k A \to V \otimes_k A$, or simply the $n \times n$ matricies with A entries. This is represented by

$$k[\{x_{ij} : 1 \leq i,j \leq n\}][y]/(ydet(x_{ij})-1)$$

A ***representation*** of an affine group scheme (from now on GS) G is a morphism (natural transformation) of GS

$$G \to GL_V$$

We say that a representation is faithful if the associated map

$$\mathcal{O}(GL_V) \to \mathcal{O}(G)$$

is surjective

would usually make more sense to be injective. Is it because we are in the opposite cat or something

A GS is ***linear*** if there exists some faithful representation.

It is of ***finite type*** if it is represented by a finitely generated algebra.

An ***affine algebraic group*** (AAG) is an GS of finite type over a field.

Theorem. *All AAG are linear.*

Let G be a GS then G is

- ***connected*** iff the only idempotents of $\mathcal{O}(G)$ are 0 and 1
- ***reduced*** iff $\mathcal{O}(G)$ has no (non-zero) nilpotent elements
- ***irreducible*** iff the nilradical (collection of nilpotent elements) of $\mathcal{O}(G)$ is a prime ideal
- ***integral*** iff $\mathcal{O}(G)$ is an integral domain
- ***smooth*** iff $\mathcal{O}(G)$ is formally smooth

Lemma. *If $k \subseteq \mathbb{C}$ then G is connected iff $G(\mathbb{C})$ is connected in the analytic topology.*

A ***subgroup scheme*** $H \subseteq G$ is a subscheme (not defined here) such that on points $H(A) \subseteq G(A)$ we have subgroups.

If G is a AAG then there is a subgroup scheme G° such that G° is

- normal
- contains the identity
- maximally connected

G° is called the ***neutral component***

is it unique or are they conjugate?

Recall that an element $x \in M_n(k)$ is called

- ***semisimple*** if there is some $g \in M_n(\bar{k})$ such that $g^{-1}xg$ is diagonal
- ***nilpotent*** if there is some $n \in \mathbb{N}$ such that $x^n = 0$
- ***unipotent*** if $x - I$ is nilpotent

Theorem. *If G is a linear GS then an element $r \in G(R)$ is —-one of the things above—- if there exists a faithful representation of G, ρ such that $\rho(r)$ —-is that thing—-.*

An AAG is called ***unipotent*** if every representation has a fixed vector.

The ***derived subgroup*** of an AAG G is the intersection of all normal subgroups (normal on points) $N \subseteq G$ such that G/N is commutative. We denote this G^{det} or $\mathcal{D}G$.

We say G is ***solvable*** if there is some $n \in \mathbb{N}$ such that $\mathcal{D}^n(G)$ is trivial.

The ***unipotent radical*** of G, denoted $R_U(G)$, is the maximal connected, normal and unipotent subgroup. The ***solvable radical*** of G, denoted $R(G)$, is the maximal connected, normal and solvable subgroup.

A smooth, connected AAG G is ***reductive*** iff $R_U(G_{\bar{k}}) = \{1\}$ and semi-simple iff $R(G_{\bar{k}}) = \{1\}$.

A subgroup $M \subseteq G$ of an AAG is a ***Levi subgroup*** iff the following is exact

$$1 \to M_{\bar{k}} \hookrightarrow G_{\bar{k}} \xrightarrow{\pi} G_{\bar{k}}/R_U(G_{\bar{k}}) \to 1$$

An ***algebraic torus*** is an AAG T such that for some $n \in \mathbb{N}$ $T_{\bar{k}} \cong \mathbb{G}_m^n$. We say that T ***splits*** (as a torus) if $T \cong \mathbb{G}_m^n$. T is a ***maximul torus*** in G if $T \subseteq G$ and $T_{\bar{k}}$ is a maximal element of the set of tori (ordered by inclusion).

If $T \subseteq G$ is a torus inside a reductive group then $N_G(T)$ is the normalizer and $C_G(T)$ is its centralizer. The Weyl group is $W(G,T) = N_G(T)/C_G(T)$.

Lemma. *T is a maximal torus iff $C_G(T) = T$.*

A reductive group is ***split*** if there exists a split maximal torus.

If G is reductive then $B \subseteq G$ is called a ***Borel*** iff $B_{\bar{k}}$ is a maxima, smooth, connected and solvable subgroup of $G_{\bar{k}}$. A smooth subgroup $P \subseteq G$ is ***parabolic*** if $P_{\bar{k}}$ contains a borel subgroup of $G_{\bar{k}}$.

A reductive group is ***quasi-split*** if it contains a Borel. If G contains a split torus it is called ***isotropic***.

Given a minimal parabolic subgroup $P_0 \subseteq G$ then the parabolic subgroups that contain P_0 are called ***standard***.

1 Working out GL_2

We used [**?**] for some help on the examples. Many theorems again from [Mil]. I will try to make explicit what I am and am not "black boxing" (taking as axioms).

It is clear that GL_2 is an affine algebraic group we further claim that

represented by ...

Axiom. *GL_2 is reductive.*

1.1 Borels and Tori

We will check that some things are Borels and Tori, as of now we have no systematic way of finding all of them (they are conjugate over an algebraically closed field.) .

systamatize

Lemma.

$$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

is a borel.

Axiom. *B is an affine group scheme that is smooth and connected.*

Theorem. *Pseudo-reductive groups by Conrad, Gabber and Prasad (Definition A.1.14, lightly edited)*

The derived group $D(G)$ of a smooth group G of finite type over a field k is the unique smooth closed k-subgroup such that $(D(G))(K)$ is the commutator subgroup of $G(K)$ for any algebraically closed extension K/k.

Proof (Of Lemma)**.** *What we will show is that it is maximal solvable. To see that it is solvable we will look at its points for K/k an algebraicly closed extension.*

have better proof environment in minipreable.

$$D(B)(K) = [B,B](K) = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$$

by simply computing the commutator of two arbitrary elements of B.

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} e & f \\ 0 & h \end{pmatrix}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^{-1}\begin{pmatrix} e & f \\ 0 & h \end{pmatrix}^{-1} = \begin{pmatrix} 1 & af+bh-df-eb \\ 0 & 1 \end{pmatrix}$$

Then by substituting in $a, d, e, h = 1$ we get that

$$\begin{pmatrix} 1 & af+bh-df-eb \\ 0 & 1 \end{pmatrix} = I$$

Hence

$$D^2(B)(K) = \{e\}$$

So B is solvable.

To see that it is maximal: If B is contained in a subgroup $B \neq B' \subseteq GL_2$ then at least B' has one element with non-zero bottom left entry say $a \in K$ (for its K points), and therefore because it is a subgroup we know that

$$\langle a \rangle \quad := \quad \langle \begin{pmatrix} * & * \\ a & * \end{pmatrix} \rangle \subseteq B'$$

Now we claim that $\langle a \rangle = GL_2$. I think if you multiply with an arbitrary element you will get something in the bottom left that depends on all the variables and you will be able to make the entries anything you want.

prove this.

The final fact that we need is that GL_2 is not solvable.

$$D(GL_2)(K) = SL_2(K)$$

by comparing the determinants we know that $[GL_2, GL_2] \subseteq SL_2$ and

Axiom. $SL_2 \subseteq [GL_2, GL_2]$

Again we think its a calculation that can be done..

and

$$[SL_2, SL_2](K) = SL_2(K)$$

becuase

Axiom. *SL_2 has no (non-trivial) normal subgroups.*

□

Lemma.

$$T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$$

is a Torus

Proof. *In fact it is a split torus because it is clearly iso to $\mathbb{G}_m^2$ (neither of the diagonals can be zero or it wont be invertible).*

1.2 Root Datum

The characters and co-characters of T above can be gotten from general theory [Mil, XIV.4.8]

Theorem. *If $\mathbb{G} = Hom(k[G], -)$ where $k[G]$ is the group algebra of an abelian group G then $X^*(\mathbb{G}) = G$.*

Which for us works out to mean that

Lemma. *$X^*(T) \cong \mathbb{Z}^2$ via*

$$\left[\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \mapsto t_1^{k_1} t_2^{k_2}\right] \mapsto (k_1, k_2)$$

Note that $X^*(T) \cong \mathbb{Z}^2$ becuase the natural transformations are specified by the maps written down (all of the constituent maps are of this form).

For co-characters [?] states that we also have for a split torus that $X_*(T) \cong \mathbb{Z}^{rank(T)}$,

find a reference for the proof, not in Milne

Lemma. *$X_*(T) \cong \mathbb{Z}^2$ via*

$$(k_1, k_2) \mapsto \left[t \mapsto \begin{pmatrix} t^{k_1} & 0 \\ 0 & t^{k_2} \end{pmatrix}\right]$$

Which as long as we beleive the above is valid, by checking that these are infact natrual and that they are linearly independent (hence span the free $\mathbb{Z}$ module).

Axiom. *$Lie(GL_2) = M_2(k)$*

check this, in particular the field its over

Then we want to find the roots, so we will have to decompose the following action

$$Ad : GL_2 \to GL_{LieGL_2}$$

$$GL_2(R) \to GL_{M_2(k)}(R) \quad := \quad M_2(k) \otimes_k R$$

Axiom.

$$M_2(k) \otimes_k R \cong M_2(R)$$

Where $g.m = gmg^{-1}$, as R valued matricies. Then the weight spaces are [Int]

$$\mathfrak{g}_\alpha \otimes_k R \quad := \quad \{X \in M_n(R) : \forall t \in T(R)\ Ad(t).X = \alpha(t)X\}$$

That is, the weight spaces are the set of "eigenvectors" of the adjoint action with "eigenvalue" α.

We have a basis for our roots above and so we will compute an arbitrary weight space by solving the equation

$$\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}^{-1} = \begin{pmatrix} a & t_1 t_2^{-1} b \\ t_2 t_1^{-1} c & d \end{pmatrix} = t_1^{k_1} t_2^{k_2} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Where $t_i \in R^\times$ and at least one $k_i \neq 0$. To find the roots we need $\mathfrak{g}_\alpha \otimes_k R \neq 0$ for all R so it is sufficient to consider an algebraically closed extension of K/k. In particular K is infinite and $K^\times = K - 0$. Thus the condition above implies that

$$a = d = 0$$

because multiplication by a unit gives a bijection from $K \to K$. Rearanging we require

$$c = t_1^{k_1+1} t_2^{k_2-1} c, \quad b = t_1^{k_1-1} t_2^{k_2+1} b$$

These are mutually exclusive, in the sense that if $k_1 = 1, k_2 = -1$ there is no condition on b and hence it can be anything but it forces c to be zero. Likewise for $k_1 = -1, k_2 = 1$ freeing c but enforcing $b = 0$.

Thus the roots are $(-1, 1)$ and $(1, -1)$ with respective root spaces

$$\begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$$

Now for the corrots. We require (by definition) that the coroots satisfy

$$\langle \alpha, \alpha^\vee \rangle \ := \ \alpha^\vee \circ \alpha = 2$$

This plus the ansatz from Milne or elsewhere makes this easy. Whats hard is understanding where this inner product came from, I mean its given as a definition in some but in Getz and Hahn they say something crazy.

2 Working out Sp_{2n}

The title might be a little bit grandiose, I will probably only do Sp_4.

References

[Int] *An Introduction to Automorphic Representations*.

[Mil] J S Milne. Basic Theory of Affine Group Schemes.